# 39．非线性方程组求解能力对比测试 — 1stOpt v.s. Matlab、Mathematica 及 Maple

39.1 引言

方程组求解计算是科学数值计算最基础同时也是最重要的组成部分之一，很多科学计算问题最终都转化为方程组计算问题。对于线性方程组，相对简单，有高效可靠的算法能保证得到正解，但对于非线性方程组，仍然面临巨大挑战，没有任何一种算法或软件工具能保证一定得到正解。

科学数值计算软件，最有名的莫过于 Matlab、Mathematica 及 Maple，俗称"3M"数学软件三剑客（ 包括 MathCAD，之前也有"4M"之说）。"3M"软件中，Matlab 最擅长数值计算，使用也最为广泛，而 Mathematica 和 Maple 以符号推理计算见长，同时也兼具数值计算。三款软件都内置了非线性方程组数值解计算功能及相关命令，如 Matlab 的"fsolve"和"vpasolve"，Mathematica 的"NSolve"和"FindRoot"以及 Maple 的"fsolve"。

非线性方程组求解也是 1stOpt 重要的功能组成之一。"3M"软件的非线性方程组计算效果如何？使用是否方便？与 1stOpt 相比孰优孰劣？在此基于十二道非线性方程组实际案例问题进行初步对比验证。
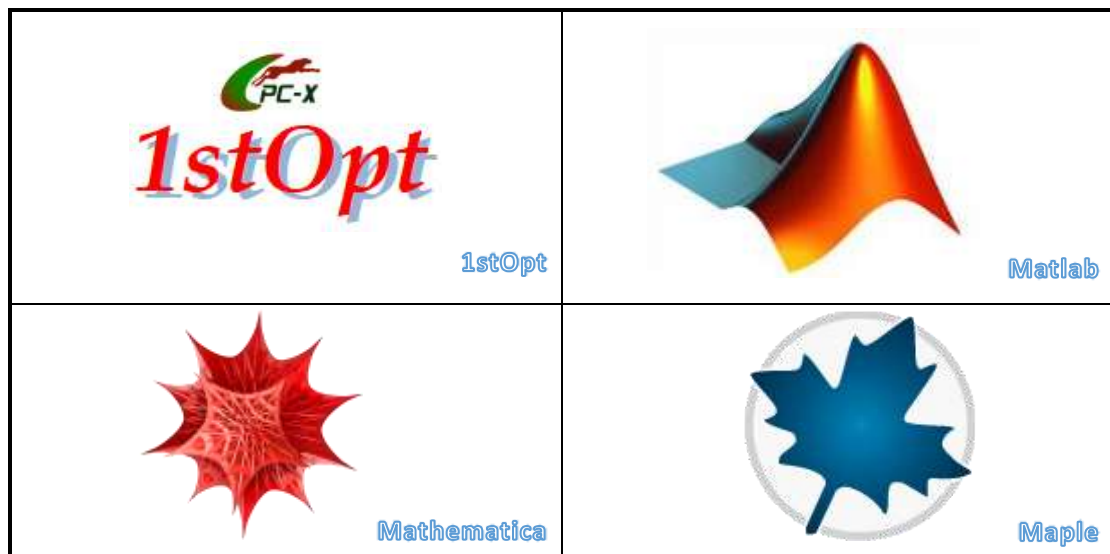
39.2 案例及计算对比测试

选用十二道实际应用中遇到的非线性方程组，这些方程组看上去都极其普通，最多包含常用的指数及对数函数，不涉及任何特殊函数甚至三角函数，求解维度也不高，从三维到八维不等。

"3M"软件中，仅测试 Matlab 的"fsolve"，Mathematica 的"FindRoot"以及 Maple 的"fsolve"。由于这些命令的计算结果均与初始值有关，因此初始值均采用[-10,10]之间的随机值。1stOpt 不需人为赋初值而由软件本身内部自动随机产生。四款软件都各自独立运行 10 次，计算结

果是否为正解由方程误差精度判断，一般要求方程误差（每个方程左右相减值）达到 1E-10 的级别。

39.2.1 案例-1：三元三次方程组

$$\begin{cases} (1-x_1)\cdot 0.7133-(1-0.7133)=x_1\cdot\left(1-\exp\left(-\frac{(1-x_1)\cdot 127}{x_2}\right)\right)\cdot\exp\left(-\frac{127}{x_3}\right) \\ (1-x_1)\cdot 0.8058+(1-0.8058)=x_1\cdot\left(1-\exp\left(-\frac{(1-x_1)\cdot 229}{x_2}\right)\right)\cdot\exp\left(-\frac{229}{x_3}\right) \qquad (39\text{-}1) \\ (1-x_1)\cdot 0.7133-(1-0.8708)=x_1\cdot\left(1-\exp\left(-\frac{(1-x_1)\cdot 421}{x_2}\right)\right)\cdot\exp\left(-\frac{421}{x_3}\right) \end{cases}$$

案例-1 代码

1stOpt	Hardness = 2; Function (1-x1)*0.7133-(1-0.7133)= x1*(1-exp(-(1-x1)*127/x2))*exp(-127/x3); (1-x1)*0.8058+(1-0.8058)= x1*(1-exp(-(1-x1)*229/x2))*exp(-229/x3); (1-x1)*0.7133-(1-0.8708)= x1*(1-exp(-(1-x1)*421/x2))*exp(-421/x3);

Matlab	equations = @(x) [(1-x(1))*0.7133-(1-0.7133)-(x(1)*(1-exp(0-(1-x(1))*127/x(2)))*exp(0-127/x(3)));
	(1-x(1))*0.8058+1-0.8058-(x(1)*(1-exp(0-(1-x(1))*229/x(2)))*exp(0-229/x(3)));
	(1-x(1))*0.7133-(1-0.8708)-(x(1)*(1-exp(0-(1-x(1))*421/x(2)))*exp(0-421/x(3)))];
	InitialGuess = 10*(2*rand(3,1)-1); %[1,1,1];
	%InitialGuess = ones(3,1); %[1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{(1-x1)*0.7133-(1-0.7133)-(x1*(1-Exp[0-(1-x1)*127/x2])*Exp[0-127/x3]) == 0, (1-x1)*0.8058+1-0.8058-(x1*(1-Exp[0-(1-x1)*229/x2])*Exp[0-229/x3]) == 0, (1-x1)*0.7133-(1-0.8708)-(x1*(1-Exp[0-(1-x1)*421/x2])*Exp[0-421/x3]) == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({(1-x1)*0.7133-(1-0.7133)-(x1*(1-exp(0-(1-x1)*127/x2))*exp(0-127/x3)) = 0, (1-x1)*0.8058+1-0.8058-(x1*(1-exp(0-(1-x1)*229/x2))*exp(0-229/x3)) = 0, (1-x1)*0.7133-(1-0.8708)-(x1*(1-exp(0-(1-x1)*421/x2))*exp(0-421/x3)) = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1)})

案例-1 结果：

	参数值	方程误差值
1stOpt	x1: 24.6585100975876 x2: 2002.68074188802 x3: 78.8665346900753	-7.105427357601E-15 3.5527136788005E-15 -3.5527136788005E-15
Matlab	要么无解，要么非正解，如： x1 = 0.650398789596784 x2 = 83.6695142051881 x3 = 400.245114026375	-0.232331524448736 0.249857694149897 -0.0678863445966185
Mathematica	每次运行结果都不同，都是与正解相差很远的近似解，如： x1 = -2.13222 x2 = 9.4542 x3 = 9.43613	1.94751557180988 2.71814287606154 2.105012526
Maple	无解或错误结果，如：	

x1 = -3.8586935440000000	-9.07531432126194E40
x2 = -8.0786647770000000	-2.41075384736122E73
x3 = -7.6607866820000000	-2.60683030432902E134

"3M"软件均无法得到正解；注意 1stOpt 代码中加了一句"Hardness = 2;"，表明该问题的难度较大，由缺省的难度"1"变为"2"（共划分 1 至 5 个难度），程序会因此自动调整求解控制参数，增加得到最优解的概率；在缺省难度为"1"时求得最优解的概率为 30 左右，变为"2"后，成功概率变为 90%。

39.2.2 案例-2 三元三次方程组

$$\begin{cases} x_1^{x_2} + x_2^{x_1} - 5 \cdot x_1 \cdot x_2 \cdot x_3 - 85 = 0 \\ x_1^3 \cdot x_2^{x_3} \cdot x_3^{x_2} - 60 = 0 \\ x_1^{x_3} + x_3^{x_1} - x_2 - 0.55 = 0 \end{cases} \tag{39-2}$$

案例-2 代码

1stOpt	Function x1^x2+x2^x1-5*x1*x2*x3-85=0;
	　　　　　x1^3*x2^x3*x3^x2-60=0;
	　　　　　x1^x3+x3^x1-x2-0.55=0;

Matlab	equations = @(x) [x(1)^x(2)+x(2)^x(1)-5*x(1)*x(2)*x(3)-85-0;
	x(1)^3*x(2)^x(3)*x(3)^x(2)-60-0;
	x(1)^x(3)+x(3)^x(1)-x(2)-0.55-0];
	InitialGuess = 10*(2*rand(3,1)-1);　　%[1,1,1];
	%InitialGuess = ones(3,1);　　%[1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{x1^x2+x2^x1-5*x1*x2*x3-85-0 == 0, x1^3*x2^x3*x3^x2-60-0 == 0, x1^x3+x3^x1-x2-0.55-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({x1^x2+x2^x1-5*x1*x2*x3-85-0 = 0, x1^3*x2^x3*x3^x2-60-0 = 0, x1^x3+x3^x1-x2-0.55-0 = 0}, {x1=10*(2*Generate(float)-1),　x2=10*(2*Generate(float)-1),　x3=10*(2*Generate(float)-1)})

案例-2 结果：

	参数值	方程误差
1stOpt	x1: 19390.8841948434 x2: 0.45 x3: 2.33057529481649E-25	-3.26849658449646E-13 1.30029320644098E-12 -3.99680288865056E-15
Matlab	要么无解，要么只得到复数解	
Mathematica	要么无解，要么只得到复数解	
Maple	无法得到正确实数解	

只有 1stOpt 可以得到正解，缺省状态下 20%的成功率，如果加上"Hardness = 2;"，成功率将升至 50%。此道题虽然只有区区 3 个参数，但求解难度很大，参考 1stOpt 的参数计算

结果，即使给三款"3M"软件赋予很接近最优值的初值，如 x=[20000,1,0]，也仍然无法得到正解，而如果给 1stOpt 赋予这样的初值，成功率将为 100%。

39.2.3 案例-3：三元三次方程组

$$\begin{cases} 2 \cdot x_1 \cdot \left(1 - x_2 \cdot \exp(-13 \cdot x_3)\right) = 2.07 \\ 3 \cdot x_1 \cdot \left(1 - x_2 \cdot \exp(-21 \cdot x_3)\right) = 3.22 \\ 4 \cdot x_1 \cdot \left(1 - x_2 \cdot \exp(-29 \cdot x_3)\right) = 5.14 \end{cases} \tag{39-3}$$

案例-3 代码

1stOpt	Parameter x1,x2,x3; Function 2*x1*(1-x2*exp(-13*x3))=2.07; 　　　　3*x1*(1-x2*exp(-21*x3))=3.22; 　　　　4*x1*(1-x2*exp(-29*x3))=5.14;

Matlab	equations = @(x) [2*x(1)*(1-x(2)*exp(0-13*x(3)))-2.07;
	3*x(1)*(1-x(2)*exp(0-21*x(3)))-3.22;
	4*x(1)*(1-x(2)*exp(0-29*x(3)))-5.14];
	InitialGuess = 10*(2*rand(3,1)-1);　　%[1,1,1];
	%InitialGuess = ones(3,1);　　%[1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{2*x1*(1-x2*Exp[0-13*x3])-2.07 == 0, 3*x1*(1-x2*Exp[0-21*x3])-3.22 == 0, 4*x1*(1-x2*Exp[0-29*x3])-5.14 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({2*x1*(1-x2*exp(0-13*x3))-2.07 = 0, 3*x1*(1-x2*exp(0-21*x3))-3.22 = 0, 4*x1*(1-x2*exp(0-29*x3))-5.14 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1)})

案例-3 结果：

求解器	参数值	方程误差值
1stOpt	x1: 1.02652243590673 x2: -0.000514078360857846 x3: -0.21358660881644	1.78586034849104E-11 2.21276330591991E-11 -3.03845837379413E-12
Matlab	要么无解，要么只能给出近似实数解，如： x1 = 1666.77674862914 x2 = 0.999556379472030 x3 = 1.09984162407442e-05	 -0.114783221524789 0.152513479556321 -0.0571158585310094
Mathematica	要么无解，要么只能给出近似实数解，如： 5.386756645619156*10^-59 -4.331007849135506 -4.584604671108670	 -2.07 -3.22 1.68753899743024E-14
Maple	x1 = 1.0265224360000000 x2 = -0.0005140783609000 x3 = -0.2135866088000000	2.03705941004273E-10 2.7773028321576E-10 5.58344481760287E-11

极其平常普通的低维度非线性方程组，但 Matlab 和 Mathematica 最好仅能给出近似解；1stOpt 和 Maple 均能以 100%概率得到正解。

39.2.4 案例-4：三元三次方程组

$$\begin{cases} \dfrac{x_1}{1 + x_2 \cdot (2.72^{-x_3 \cdot 1990})} - \dfrac{x_1}{1 + x_2} = 0.22 \\[2mm] \dfrac{x_1}{1 + x_2 \cdot (2.72^{-x_3 \cdot 2002})} - \dfrac{x_1}{1 + x_2} = 0.31 \\[2mm] \dfrac{x_1}{1 + x_2 \cdot (2.72^{-x_3 \cdot 2005})} - \dfrac{x_1}{1 + x_2} = 0.33 \end{cases} \qquad （39\text{-}4）$$

案例-4 代码

1stOpt	```
	Function
	x1/(1+x2*(2.72^(-x3*1990)))-x1/(1+x2)=0.22;
	x1/(1+x2*(2.72^(-x3*2002)))-x1/(1+x2)=0.31;
	x1/(1+x2*(2.72^(-x3*2005)))-x1/(1+x2)=0.33;
	``

Matlab	```
	equations = @(x) [x(1)/(1+x(2)*(2.72^(-x(3)*1990)))-x(1)/(1+x(2))-0.22;
	x(1)/(1+x(2)*(2.72^(-x(3)*2002)))-x(1)/(1+x(2))-0.31;
	x(1)/(1+x(2)*(2.72^(-x(3)*2005)))-x(1)/(1+x(2))-0.33];
	InitialGuess = 10*(2*rand(3,1)-1); %[1,1,1];
	%InitialGuess = ones(3,1); %[1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);
	``

Mathematica	``` FindRoot[{x1/(1+x2*(2.72^(-x3*1990)))-x1/(1+x2)-0.22 == 0, x1/(1+x2*(2.72^(-x3*2002)))-x1/(1+x2)-0.31 == 0, x1/(1+x2*(2.72^(-x3*2005)))-x1/(1+x2)-0.33 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16] ```
Maple	``` interface(displayprecision = 16) with(RandomTools): fsolve({x1/(1+x2*(2.72^(-x3*1990)))-x1/(1+x2)-0.22 = 0, x1/(1+x2*(2.72^(-x3*2002)))-x1/(1+x2)-0.31 = 0, x1/(1+x2*(2.72^(-x3*2005)))-x1/(1+x2)-0.33 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1)}) ```

案例-4 结果：

求解器	参数值	方程误差值
1stOpt	第一组： x1: -0.466327667279565 x2: 3.188084562311E-58 x3: -0.0664291202364013 第二组： x1: 0.466327667302067 x2: 3.13667962832981E57 x3: 0.0664291202313272	第一组： 1.37212463613423E-12 2.09482431401398E-12 1.37584388326673E-12 第二组： -3.24740234702858E-15 -3.38618022510673E-15 -2.60902410786912E-15
Matlab	无正解	
Mathematica	无正解	
Maple	无正解	

  本题根据 1stOpt 的计算结果看，有两组"对称"的解，其特点是参数 x2 数值要么特别小，要么又特别大，虽然只有三个未知数，但对"3M"软件而言却"难于上青天"，即使给出接近最优值的初值如 x=[-0.5, 1e-55, -0.1]也无法给出正解。

39.2.5 案例-5：四元四次方程组

$$x_1 + x_2 \cdot \exp\left(\frac{x_3}{a_i}\right) + x_4 \cdot a_i - b_i = 0, \qquad i = 1..4 \qquad （39-5）$$

其中：a=[1423.59,1430.57,1449.69,1481.98], b=[21.95,11.18,7.69,3]

案例-5 代码

1stOpt	Constant a=[1423.59,1430.57,1449.69,1481.98],b=[21.95,11.18,7.69,3]; Function For(a,b)(x1+x2*exp(x3/a)+x4*a-b=0);

Matlab	equations = @(x) [x(1)+x(2)*exp(x(3)/1423.59)+x(4)*1423.59-21.95-0;
	x(1)+x(2)*exp(x(3)/1430.57)+x(4)*1430.57-11.18-0;
	x(1)+x(2)*exp(x(3)/1449.69)+x(4)*1449.69-7.69-0;
	x(1)+x(2)*exp(x(3)/1481.98)+x(4)*1481.98-3-0];
	InitialGuess = 10*(2*rand(4,1)-1); %[1,1,1,1];
	%InitialGuess = ones(4,1); %[1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{x1+x2*Exp[x3/1423.59]+x4*1423.59-21.95-0 == 0, x1+x2*Exp[x3/1430.57]+x4*1430.57-11.18-0 == 0, x1+x2*Exp[x3/1449.69]+x4*1449.69-7.69-0 == 0, x1+x2*Exp[x3/1481.98]+x4*1481.98-3-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({x1+x2*exp(x3/1423.59)+x4*1423.59-21.95-0 = 0, x1+x2*exp(x3/1430.57)+x4*1430.57-11.18-0 = 0, x1+x2*exp(x3/1449.69)+x4*1449.69-7.69-0 = 0, x1+x2*exp(x3/1481.98)+x4*1481.98-3-0 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1)})

案例-5 结果：

求解器	参数值	方程误差值
1stOpt	x1: 218.228110910674 x2: 9.10236261977445E-239 x3: 783626.245975248 x4: -0.145230104937087	-1.94333438230387E-12 -1.01607611213694E-12 -7.13207271019201E-13 -7.95807864051312E-13
Matlab	只能给出相差很远的近似实数解，如： x1 = 7.59305893673008 x2 = -7.95679582994413 x3 = -0.995505844279068 x4 = 0.00775403955282771	-11.2696015433881 -0.445505476725685 3.19267878262024 8.13293774974168
Mathematica	无法给出正解	
Maple	无法给出正解	

这是一道非常有特色的方程组，参考 1stOpt 得出的结果，其中一个参数 x2 达到了 1E-239 的程度但却又不能为 0，如下表验证对比结果，很好体现了"失之毫厘谬以千里"的现象。"3M"软件，如果说在[-10,10]之间随机初值无法得到正解实属正常的话，即使给出非常接近最优值的初值，如 x=[200, 1e-235, 800000, -0.15]（虽然人工猜出这样的初值几无可能），但仍无法获得正解，令人多少有些小小的失望加吃惊。

x2 取值	方程误差值
9.10236262007253E-239	2.43360886997834E-12 1.49213974509621E-13 8.26005930321116E-14 1.13686837721616E-13
9.10236261977445E-238	94.2301275904448 6.42348278256942 0.00467924014236321 3.59085277068516E-8
9.10236261977445E-240	-9.42301275904178 -0.642348278256769 -0.000467924014148302 -3.59074192601838E-9
0	-10.4700141767134 -0.713720309174214 -0.000519915571291385 -3.98972588300239E-

39.2.6 案例-6: 四元四次方程组

$$\begin{cases} 1 - \exp(-(x_1 + x_2 \cdot 915) \cdot 1200^{x_3 + x_4 \cdot 915}) = \dfrac{1.600}{100} \\ 1 - \exp(-(x_1 + x_2 \cdot 909) \cdot 1212^{x_3 + x_4 \cdot 909}) = \dfrac{3.540}{100} \\ 1 - \exp(-(x_1 + x_2 \cdot 898) \cdot 1234^{x_3 + x_4 \cdot 898}) = \dfrac{5.430}{100} \\ 1 - \exp(-(x_1 + x_2 \cdot 895) \cdot 1240^{x_3 + x_4 \cdot 895}) = \dfrac{5.683}{100} \end{cases} \quad （6）$$

案例-6 代码

1stOpt	Function 1-exp(-(x1+x2*915)*1200^(x3+x4*915))=1.600/100; 1-exp(-(x1+x2*909)*1212^(x3+x4*909))=3.540/100; 1-exp(-(x1+x2*898)*1234^(x3+x4*898))=5.430/100; 1-exp(-(x1+x2*895)*1240^(x3+x4*895))=5.683/100;

Matlab	equations = @(x) [1-exp(0-(x(1)+x(2)*915)*1200^(x(3)+x(4)*915))-(1.600/100);
	1-exp(0-(x(1)+x(2)*909)*1212^(x(3)+x(4)*909))-(3.540/100);
	1-exp(0-(x(1)+x(2)*898)*1234^(x(3)+x(4)*898))-(5.430/100);
	1-exp(0-(x(1)+x(2)*895)*1240^(x(3)+x(4)*895))-(5.683/100)];
	InitialGuess = 10*(2*rand(4,1)-1); %[1,1,1,1];
	%InitialGuess = ones(4,1); %[1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{1-Exp[0-(x1+x2*915)*1200^(x3+x4*915)]-(1.600/100) == 0, 1-Exp[0-(x1+x2*909)*1212^(x3+x4*909)]-(3.540/100) == 0, 1-Exp[0-(x1+x2*898)*1234^(x3+x4*898)]-(5.430/100) == 0, 1-Exp[0-(x1+x2*895)*1240^(x3+x4*895)]-(5.683/100) == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools):

	fsolve({1-exp(0-(x1+x2*915)*1200^(x3+x4*915))-(1.600/100) = 0, 1-exp(0-(x1+x2*909)*1212^(x3+x4*909))-(3.540/100) = 0, 1-exp(0-(x1+x2*898)*1234^(x3+x4*898))-(5.430/100) = 0, 1-exp(0-(x1+x2*895)*1240^(x3+x4*895))-(5.683/100) = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1)})

案例-6 结果：

求解器	参数值	方程误差值
1stOpt	x1: 180269.749976749 x2: -196.244818619846 x3: -4.99648566803081 x4: 0.00381339657343384	-2.08166817117217E-16 3.19189119579733E-16 1.13103970633688E-15 9.36750677027476E-16
Matlab	无法给出正解	
Mathematica	无法给出正解，只有近似解，如： 4.053004656357142 2.842659487302639 -2.402054449041441 -0.07449075855549998	-0.016 -0.0354 -0.0543 -0.05683
Maple	无法给出正解	

三款"3M"软件均无法得到正确结果，而 1stOpt 能以仅 100%概率求出正解。

39.2.7 案例 7：四元四次方程组

$$\begin{cases} 6250 \cdot \pi \cdot x_1 + 2500 \cdot \pi \cdot x_1 \cdot (15 - x_2) + 10.5 - x_3 = 0 \\ -169169 \cdot x_2 + 168971 \cdot x_2 - x_4 \cdot x_3 - 1171.7 = 0 \\ 1.03 \cdot x_2 + 1.84 \cdot x_2 \cdot x_4 \cdot x_3 + 2500 \cdot x_4 \cdot x_3 - x_3 = 0 \\ 6166 - 282 \cdot x_2 - x_1 \cdot (20 - x_2) = 0 \end{cases} \tag{7}$$

案例-7 代码

1stOpt	Hardness = 2; //1,2,...5 Function 6250*pi*x1+2500*pi*x1*(15-x2)+10.5-x3=0; -169169*x2+168971*x2-x4*x3-1171.7=0; 1.03*x2+1.84*x2*x4*x3+2500*x4*x3-x3=0; 6166-282*x2-x1*(20-x2)=0;

Matlab	equations = @(x) [6250*pi*x(1)+2500*pi*x(1)*(15-x(2))+10.5-x(3)-0;
	-169169*x(2)+168971*x(2)-x(4)*x(3)-1171.7-0;
	1.03*x(2)+1.84*x(2)*x(4)*x(3)+2500*x(4)*x(3)-x(3)-0;
	6166-282*x(2)-x(1)*(20-x(2))-0];
	InitialGuess = 10*(2*rand(4,1)-1); %[1,1,1,1];
	%InitialGuess = ones(4,1); %[1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{6250*Pi*x1+2500*Pi*x1*(15-x2)+10.5-x3-0 == 0, -169169*x2+168971*x2-x4*x3-1171.7-0 == 0, 1.03*x2+1.84*x2*x4*x3+2500*x4*x3-x3-0 == 0, 6166-282*x2-x1*(20-x2)-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}},

	MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({6250*Pi*x1+2500*Pi*x1*(15-x2)+10.5-x3-0 = 0, -169169*x2+168971*x2-x4*x3-1171.7-0 = 0, 1.03*x2+1.84*x2*x4*x3+2500*x4*x3-x3-0 = 0, 6166-282*x2-x1*(20-x2)-0 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1)})

案例-7 结果：

求解器	参数值	方程误差值
1stOpt	第一组： x1: 213.02505351492 x2: 27.6259573483508 x3: -16941676.6706643 x4: 0.000392029648781691 第二组： x1: 942.493330252235 x2: 19.2036255690895 x3: -12610780.1619252 x4: 0.000394425864126739	第一组： -8.19563865661621E-8 -1.20508047984913E-11 1.11758708953857E-8 9.09494701772928E-13 第二组： -1.08033418655396E-7 -3.20596882374957E-11 1.49011611938477E-8 1.94972926692571E-10
Matlab	无正解	
Mathematica	无正解	
Maple	无正解	

　　1stOpt 能以约 100%概率得到两组解，其它三款均无法获得正解。

39.2.8 案例-8：五元五次方程组

$$\begin{cases} -x_1 - x_2 \cdot x_3 - x_4 \cdot x_5 + 12.04 = 0 \\ x_2 \cdot x_3^2 - x_4 \cdot x_5^2 - 20.70 = 0 \\ -1.30 \cdot x_1 - x_2 \cdot \ln(1 + 1.30 \cdot x_3) - x_4 \cdot (\exp(1.30 \cdot x_5) - 1) - 4.66 = 0 \\ -x_1 - \dfrac{x_2 \cdot x_3}{1 + 1.30 \cdot x_3} - x_4 \cdot x_5 \cdot \exp(1.30 \cdot x_5) - 52.57 = 0 \\ \dfrac{x_2 \cdot x_3^2}{(1 + 1.30 \cdot x_3)^2} - x_4 \cdot x_5^2 \cdot \exp(1.30 \cdot x_5) - 109.83 = 0 \end{cases} \quad (8)$$

案例-8 代码

1stOpt	Function -x1-x2*x3-x4*x5+12.04=0;
	　　　　x2*x3^2-x4*x5^2-20.70=0;
	　　　　-1.30*x1-x2*ln(1+1.30*x3)-x4*(exp(1.30*x5)-1)-4.66=0;
	　　　　-x1-x2*x3/(1+1.30*x3)-x4*x5*exp(1.30*x5)-52.57=0;
	　　　　x2*x3^2/(1+1.30*x3)^2-x4*x5^2*exp(1.30*x5)-109.83=0;

Matlab	equations = @(x) [-x(1)-x(2)*x(3)-x(4)*x(5)+12.04-0;
	x(2)*x(3)^2-x(4)*x(5)^2-20.70-0;
	-1.30*x(1)-x(2)*log(1+1.30*x(3))-x(4)*(exp(1.30*x(5))-1)-4.66-0;
	-x(1)-x(2)*x(3)/(1+1.30*x(3))-x(4)*x(5)*exp(1.30*x(5))-52.57-0;
	x(2)*x(3)^2/(1+1.30*x(3))^2-x(4)*x(5)^2*exp(1.30*x(5))-109.83-0];
	InitialGuess = 10*(2*rand(5,1)-1);　%[1,1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{-x1-x2*x3-x4*x5+12.04-0 == 0,

	x2*x3^2-x4*x5^2-20.70-0 == 0, -1.30*x1-x2*Log[1+1.30*x3]-x4*(Exp[1.30*x5]-1)-4.66-0 == 0, -x1-x2*x3/(1+1.30*x3)-x4*x5*Exp[1.30*x5]-52.57-0 == 0, x2*x3^2/(1+1.30*x3)^2-x4*x5^2*Exp[1.30*x5]-109.83-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}, {x5, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({-x1-x2*x3-x4*x5+12.04-0 = 0, x2*x3^2-x4*x5^2-20.70-0 = 0, -1.30*x1-x2*log(1+1.30*x3)-x4*(exp(1.30*x5)-1)-4.66-0 = 0, -x1-x2*x3/(1+1.30*x3)-x4*x5*exp(1.30*x5)-52.57-0 = 0, x2*x3^2/(1+1.30*x3)^2-x4*x5^2*exp(1.30*x5)-109.83-0 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1), x5=10*(2*Generate(float)-1)})

案例-8 结果：

	参数值	方程误差值
1stOpt	x1: -499.250341459814 x2: 1124.66857461431 x3: 1.12507902890424 x4: 405.29540935769 x5: -1.86049648946527	6.28830321147689E-12 -9.27613541534811E-12 3.94706489714736E-12 2.97717406283482E-12 -7.105427357601E-14
Matlab	要么无解，要么只得到距正解很远的近似解，如： x1 = 10199.7734319073 x2 = -4309990.72505758, x3 = 0.00236045541019449 x4 = -12.8403157269356 x5 = 1.35952926013614	3.26427788309106 -20.9811798028529 3.33381993445832 -7.7065663411042 5.27207755255068
Mathematica	无法给出实数解	
Maple	x1 = -499.2503415000000000 x2 = 1124.6685750000000000 x3 = 1.1250790290000000 x4 = 405.2954094000000000 x5 = -1.8604964890000000	-6.11296023578234E-7 1.28575429769739E-6 -3.35487047919969E-7 -1.22951327341525E-7 7.06111364934259E-8

只有 1stOpt 和 Maple 给出正解，Matlab 和 Mathematica 均以失败告终。

39.2.9 案例-9：五元五次方程组

$$\begin{cases} 0.0448 - x_1 \cdot \left(\dfrac{x_2 - 0.0633}{x_3 - 0.0633}\right)^{x_5} = 0 \\[2mm] 0.0471 - x_1 \cdot \left(\dfrac{x_2 - 0.0333}{x_4 - 0.0333}\right)^{x_5} = 0 \\[2mm] 0.4714 - x_1 \cdot \left(\dfrac{x_2 + 0.0633}{x_3 + 0.0633}\right)^{x_5} = 0 \\[2mm] 0.5715 - x_1 \cdot \left(\dfrac{x_2 + 0.7}{0.5946 \cdot x_4 + 0.5 \cdot x_3 + 0.7}\right)^{x_5} = 0 \\[2mm] 0.5657 - x_1 \cdot \left(\dfrac{x_2 + 0.8}{x_4 + 0.8}\right)^{x_5} = 0 \end{cases} \tag{9}$$

案例-9 代码

1stOpt	Function 0.0448-x1*((x2-0.0633)/(x3-0.0633))^x5=0; 0.0471-x1*((x2-0.0333)/(x4-0.0333))^x5=0; 0.4714-x1*((x2+0.3333)/(x3+0.3333))^x5=0; 0.5715-x1*((x2+0.7)/(0.5946*x4+0.5*x3+0.7))^x5=0; 0.5657-x1*((x2+0.8)/(x4+0.8))^x5=0;

Matlab	equations = @(param) [0.0448-param(1)*((param(2)-0.0633)/(param(3)-0.0633))^param(4)-0;
	0.0471-param(1)*((param(2)-0.0333)/(param(5)-0.0333))^param(4)-0;
	0.4714-param(1)*((param(2)+0.3333)/(param(3)+0.3333))^param(4)-0;
	0.5715-param(1)*((param(2)+0.7)/(0.5946*param(5)+0.5*param(3)+0.7))^param(4)-0;
	0.5657-param(1)*((param(2)+0.8)/(param(5)+0.8))^param(4)-0];
	InitialGuess = 10*(2*rand(5,1)-1); %[1,1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[param,fval] = fsolve(equations,InitialGuess,options);
	x1 = param(1);
	x2 = param(2);
	x3 = param(3);
	x5 = param(4);
	x4 = param(5);

Mathematica	FindRoot[{0.0448-x1*((x2-0.0633)/(x3-0.0633))^x5-0 == 0, 0.0471-x1*((x2-0.0333)/(x4-0.0333))^x5-0 == 0, 0.4714-x1*((x2+0.3333)/(x3+0.3333))^x5-0 == 0, 0.5715-x1*((x2+0.7)/(0.5946*x4+0.5*x3+0.7))^x5-0 == 0, 0.5657-x1*((x2+0.8)/(x4+0.8))^x5-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x5, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({0.0448-x1*((x2-0.0633)/(x3-0.0633))^x5-0 = 0, 0.0471-x1*((x2-0.0333)/(x4-0.0333))^x5-0 = 0, 0.4714-x1*((x2+0.3333)/(x3+0.3333))^x5-0 = 0, 0.5715-x1*((x2+0.7)/(0.5946*x4+0.5*x3+0.7))^x5-0 = 0, 0.5657-x1*((x2+0.8)/(x4+0.8))^x5-0 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x5=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1)})

案例-9 结果：

求解器	参数值	方程误差值
1stOpt	x1: 0.754428363386432 x2: 0.21999193531669 x3: 0.0988142336799259 x5: -1.90235171609445 x4: 0.0767421031370608	4.26721158408583E-13 5.63757374116847E-13 6.45983266878147E-13 -1.3935519405095E-12 -7.54396545232794E-13
Matlab	要么复数解，要么相差很远的近似实数解，如： x1 = 0.754428364086299 x2 = 0.219991936713820 x3 = 0.0988142336942515 x4 = -1.90235171416157 x5 = 0.0767421028150171	5.55876879249073E-10 1.15901346159353E-9 1.57143220658185E-9 1.21197307745291E-9 1.17912191122116E-9
Mathematica	要么复数解，要么相差很远的近似实数解，如： x1 -> 3.552883986619389 x2 -> 2.960818240360410 x3 -> 0.1623061162807060 x5 -> -2.161584660196593 x4 -> 0.8826537191916343	0.0423961311745995 -0.197762126230464 0.412181687986102 0.188706129341528 -0.0588487157903473
Maple	运行时间超长，未给出正解	

　　Matlab 十次运行大概可一次得到正解，Mathematica 最好仅能给出相差很远的近似解，Maple 一直显示在运行，最终只能人为终止计算，1stOpt 则可以很轻松以 100%概率获得正解。

39.2.10 案例-10：六元六次方程组

$$\begin{cases} x_1^2 + x_3^2 - 1 = 0 \\ x_2^2 + x_4^2 - 1 = 0 \\ x_5 \cdot x_3^3 + x_6 \cdot x_4^3 - 100 = 0 \\ x_5 \cdot x_1^{x_3} + x_6 \cdot x_2^3 + 20 = 0 \\ x_5 \cdot x_1 \cdot x_3^{x_2} + x_6 \cdot x_2 \cdot x_4^2 - 5.5 = 0 \\ x_5 \cdot x_3 \cdot x_1^2 + x_6 \cdot x_4 \cdot x_2^2 + c = 0 \end{cases} \tag{10}$$

其中，c 分别取值 30，40，50 和 60

案例-10 代码

1stOpt	Parameter x(6);
	Constant c4=30;//40,50,60
	Function x1^2+x3^2-1=0;
	　　　　x2^2+x4^2-1=0;
	　　　　x5*x3^3+x6*x4^3-100=0;
	　　　　x5*x1^x3+x6*x2^3+20=0;
	　　　　x5*x1*x3^x2+x6*x2*x4^2-5.5=0;
	　　　　x5*x3*x1^2+x6*x4*x2^2+c4=0;

Matlab	equations = @(x) [x(1)^2+x(3)^2-1-0;
	x(2)^2+x(4)^2-1-0;
	x(5)*x(3)^3+x(6)*x(4)^3-100-0;
	x(5)*x(1)^x(3)+x(6)*x(2)^3+20-0;
	x(5)*x(1)*x(3)^x(2)+x(6)*x(2)*x(4)^2-5.5-0;
	x(5)*x(3)*x(1)^2+x(6)*x(4)*x(2)^2+30-0];
	InitialGuess = 10*(2*rand(6,1)-1);
	options = optimoptions('fsolve','Display','iter','MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{x1^2+x3^2-1-0 == 0, x2^2+x4^2-1-0 == 0,

	x5*x3^3+x6*x4^3-100-0 == 0, x5*x1^x3+x6*x2^3+20-0 == 0, x5*x1*x3^x2+x6*x2*x4^2-5.5-0 == 0, x5*x3*x1^2+x6*x4*x2^2+30-0 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}, {x5, 10*(2*RandomReal[]-1)}, {x6, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({x1^2+x3^2-1-0 = 0, x2^2+x4^2-1-0 = 0, x5*x3^3+x6*x4^3-100-0 = 0, x5*x1^x3+x6*x2^3+20-0 = 0, x5*x1*x3^x2+x6*x2*x4^2-5.5-0 = 0, x5*x3*x1^2+x6*x4*x2^2+30-0 = 0}, {x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1), x4=10*(2*Generate(float)-1), x5=10*(2*Generate(float)-1), x6=10*(2*Generate(float)-1)})

案例-10 结果

	C=30	C=40	C=50
1stOpt	第一组： x1: 0.240184968662544 x2: -0.815496094330282 x3: 0.970727140255448 x4: -0.578762576651357 x5: 129.845222134722 x6: 96.8346666139437 第二组： x1: 0.251817492626888 x2: 0.81977727798232 x3: 0.967774741562773 x4: 0.572682472669906 x5: 130.799940226334 x6: -98.8067228339781	第一组： x1: 0.3184348453657 x2: 0.796377176974345 x3: 0.947944750107826 x4: 0.604800291000611 x5: 154.535909765962 x6: -143.008370949507 第二组： x1: 0.296728182029873 x2: -0.788168611728958 x3: 0.95496198143653 x4: -0.615459372733601 x5: 151.767945391039 x6: 137.998739822423	第一组： x1: 0.337889417595447 x2: -0.771116171737383 x3: 0.941185816657378 x4: -0.636694471222073 x5: 176.320526639219 x6: 182.113060658241 第二组： x1: 0.369498005183499 x2: 0.783455018890371 x3: 0.929231523445676 x4: 0.621448496156835 x5: 181.940506072026 x6: -191.592472055294
Matlab	要么无解，要么只得到复数解		
Mathematica	只能得到复数解		
Maple	x1 = 0.2518174926 x2 = 0.8197772780 x3 = 0.9677747416 x4 = 0.5726824727 x5 = 130.7999402000 x6 = -98.8067228300	x1 = 0.3184348454 x2 = 0.7963771770 x3 = 0.9479447501 x4 = 0.6048002910 x5 = 154.5359098000 x6 = -143.0083709000	无正解

Matlab 和 Mathematica 都无法给出正确的实数解，而 Maple 当取 C 值大于 45 时也无法求得正解，1stOpt，无论何种情况，均能以 100%概率给出至少一个正解。

39.2.11 案例-11：六元六次方程组

$$\begin{cases} 158.49 \cdot (1.47 + x_4) = 1.47 \cdot \left(\dfrac{1.74 + x_2 \cdot x_4}{0.23} \right)^{1.19} \\[2mm] (x_1 - 26.3 \cdot x_4) \cdot (1.47 + x_4) = x_4 \cdot \left(\dfrac{1.74 + x_2 \cdot x_4}{x_2 \cdot x_3} \right)^{x_2} \\[2mm] 173.78 \cdot (1.76 + x_5) = 1.76 \cdot \left(\dfrac{2.09 + x_2 \cdot x_5}{0.23} \right)^{1.19} \\[2mm] (x_1 - 13.2 \cdot x_5) \cdot (1.76 + x_5) = x_5 \cdot \left(\dfrac{2.09 + x_2 \cdot x_5}{x_2 \cdot x_3} \right)^{x_2} \\[2mm] 186.21 \cdot (2.01 + x_6) = 2.01 \cdot \left(\dfrac{2.39 + x_2 \cdot x_6}{0.23} \right)^{1.19} \\[2mm] (x_1 - 5.4 \cdot x_6) \cdot (2.01 + x_6) = x_6 \cdot \left(\dfrac{2.39 + x_2 \cdot x_6}{x_2 \cdot x_3} \right)^{x_2} \end{cases} \tag{11}$$

案例-11 代码

1stOpt	Function 158.49*(1.47+x4)=1.47*((1.74+x2*x4)/0.23)^1.19; (x1-26.3*x4)*(1.47+x4)=x4*((1.74+x2*x4)/(x2*x3))^x2; 173.78*(1.76+x5)=1.76*((2.09+x2*x5)/0.23)^1.19; (x1-13.2*x5)*(1.76+x5)=x5*((2.09+x2*x5)/(x2*x3))^x2; 186.21*(2.01+x6)=2.01*((2.39+x2*x6)/0.23)^1.19; (x1-5.4*x6)*(2.01+x6)=x6*((2.39+x2*x6)/(x2*x3))^x2;

Matlab	equations = @(x) [158.49*(1.47+x(4))-(1.47*((1.74+x(2)*x(4))/0.23)^1.19);
	(x(1)-26.3*x(4))*(1.47+x(4))-(x(4)*((1.74+x(2)*x(4))/(x(2)*x(3)))^x(2));
	173.78*(1.76+x(5))-(1.76*((2.09+x(2)*x(5))/0.23)^1.19);
	(x(1)-13.2*x(5))*(1.76+x(5))-(x(5)*((2.09+x(2)*x(5))/(x(2)*x(3)))^x(2));
	186.21*(2.01+x(6))-(2.01*((2.39+x(2)*x(6))/0.23)^1.19);
	(x(1)-5.4*x(6))*(2.01+x(6))-(x(6)*((2.39+x(2)*x(6))/(x(2)*x(3)))^x(2))];
	InitialGuess = 10*(2*rand(6,1)-1); %[1,1,1,1,1,1];
	%InitialGuess = ones(6,1); %[1,1,1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{158.49*(1.47+x4)-(1.47*((1.74+x2*x4)/0.23)^1.19) == 0, (x1-26.3*x4)*(1.47+x4)-(x4*((1.74+x2*x4)/(x2*x3))^x2) == 0, 173.78*(1.76+x5)-(1.76*((2.09+x2*x5)/0.23)^1.19) == 0, (x1-13.2*x5)*(1.76+x5)-(x5*((2.09+x2*x5)/(x2*x3))^x2) == 0, 186.21*(2.01+x6)-(2.01*((2.39+x2*x6)/0.23)^1.19) == 0, (x1-5.4*x6)*(2.01+x6)-(x6*((2.39+x2*x6)/(x2*x3))^x2) == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}, {x5, 10*(2*RandomReal[]-1)}, {x6, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]
Maple	interface(displayprecision = 16) with(RandomTools): fsolve({158.49*(1.47+x4)-(1.47*((1.74+x2*x4)/0.23)^1.19) = 0, (x1-26.3*x4)*(1.47+x4)-(x4*((1.74+x2*x4)/(x2*x3))^x2) = 0, 173.78*(1.76+x5)-(1.76*((2.09+x2*x5)/0.23)^1.19) = 0, (x1-13.2*x5)*(1.76+x5)-(x5*((2.09+x2*x5)/(x2*x3))^x2) = 0, 186.21*(2.01+x6)-(2.01*((2.39+x2*x6)/0.23)^1.19) = 0, (x1-5.4*x6)*(2.01+x6)-(x6*((2.39+x2*x6)/(x2*x3))^x2) = 0},

{x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1),
x4=10*(2*Generate(float)-1), x5=10*(2*Generate(float)-1), x6=10*(2*Generate(float)-1)})

案例-11 结果：

求解器	参数值	方程误差值
1stOpt	x1: -204.64585298748	1.58473131541485E-12
	x2: 1.18367346926179	9.67418064404014E-13
	x3: 0.190905481380039	-5.12957731846342E-13
	x4: -1.46999999999985	5.33094401955481E-13
	x5: -1.75984340166528	-2.75848788255928E-13
	x6: -2.00970507729331	2.76591249903646E-13
Matlab	无法给出正解	
Mathematica	无法给出正解	
Maple	无法给出正解	

1stOpt 能以近 100%的概率得到最优解，其余三款均失败。

39.2.12 案例-12：八元八次非线性方程组

$$x_1 \cdot \exp\left(-\exp(x_2 - x_3 \cdot (a_i - x_4)^2)\right) + \frac{x_5}{1 + x_6 \cdot \exp(x_7 \cdot a_i)} + x_8 = b_i \quad i = 1\dots8 \qquad (12)$$

其中：a=[1, 2, 3, 4, 5, 6, 7, 48], b=[119, 163, 240, 395, 617, 1461, 4310, 36955]；
案例-12 代码

1stOpt	Hardness = 2; Constant a=[1,2,3,4,5,6,7,48],b=[119,163,240,395,617,1461,4310,36955]; Function For(i=1:8,a,b)(x1*exp(-exp(x2-x3*(a-x4)^2))+x5/(1+x6*exp(x7*a))+x8=b);

Matlab	equations = @(x) [x(1)*exp(0-exp(x(2)-x(3)*(1-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*1))+x(8)-119;
	x(1)*exp(0-exp(x(2)-x(3)*(2-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*2))+x(8)-163;
	x(1)*exp(0-exp(x(2)-x(3)*(3-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*3))+x(8)-240;
	x(1)*exp(0-exp(x(2)-x(3)*(4-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*4))+x(8)-395;
	x(1)*exp(0-exp(x(2)-x(3)*(5-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*5))+x(8)-617;
	x(1)*exp(0-exp(x(2)-x(3)*(6-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*6))+x(8)-1461;
	x(1)*exp(0-exp(x(2)-x(3)*(7-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*7))+x(8)-4310;
	x(1)*exp(0-exp(x(2)-x(3)*(48-x(4))^2))+x(5)/(1+x(6)*exp(x(7)*48))+x(8)-36955];
	InitialGuess = 10*(2*rand(8,1)-1); %[1,1,1,1,1,1,1,1];
	options = optimoptions('fsolve','Display','iter','MaxIterations',5000,'MaxFunEvals',5000);
	[x,fval] = fsolve(equations,InitialGuess,options);

Mathematica	FindRoot[{x1*Exp[0-Exp[x2-x3*(1-x4)^2]]+x5/(1+x6*Exp[x7*1])+x8-119 == 0, x1*Exp[0-Exp[x2-x3*(2-x4)^2]]+x5/(1+x6*Exp[x7*2])+x8-163 == 0, x1*Exp[0-Exp[x2-x3*(3-x4)^2]]+x5/(1+x6*Exp[x7*3])+x8-240 == 0, x1*Exp[0-Exp[x2-x3*(4-x4)^2]]+x5/(1+x6*Exp[x7*4])+x8-395 == 0, x1*Exp[0-Exp[x2-x3*(5-x4)^2]]+x5/(1+x6*Exp[x7*5])+x8-617 == 0, x1*Exp[0-Exp[x2-x3*(6-x4)^2]]+x5/(1+x6*Exp[x7*6])+x8-1461 == 0, x1*Exp[0-Exp[x2-x3*(7-x4)^2]]+x5/(1+x6*Exp[x7*7])+x8-4310 == 0, x1*Exp[0-Exp[x2-x3*(48-x4)^2]]+x5/(1+x6*Exp[x7*48])+x8-36955 == 0}, {{x1, 10*(2*RandomReal[]-1)}, {x2, 10*(2*RandomReal[]-1)}, {x3, 10*(2*RandomReal[]-1)}, {x4, 10*(2*RandomReal[]-1)}, {x5, 10*(2*RandomReal[]-1)}, {x6, 10*(2*RandomReal[]-1)}, {x7, 10*(2*RandomReal[]-1)}, {x8, 10*(2*RandomReal[]-1)}}, MaxIterations -> 100, WorkingPrecision -> 16]

	interface(displayprecision = 16)
Maple	with(RandomTools):
	fsolve({x1*exp(0-exp(x2-x3*(1-x4)^2))+x5/(1+x6*exp(x7*1))+x8-119 = 0,
	x1*exp(0-exp(x2-x3*(2-x4)^2))+x5/(1+x6*exp(x7*2))+x8-163 = 0,
	x1*exp(0-exp(x2-x3*(3-x4)^2))+x5/(1+x6*exp(x7*3))+x8-240 = 0,
	x1*exp(0-exp(x2-x3*(4-x4)^2))+x5/(1+x6*exp(x7*4))+x8-395 = 0,
	x1*exp(0-exp(x2-x3*(5-x4)^2))+x5/(1+x6*exp(x7*5))+x8-617 = 0,
	x1*exp(0-exp(x2-x3*(6-x4)^2))+x5/(1+x6*exp(x7*6))+x8-1461 = 0,
	x1*exp(0-exp(x2-x3*(7-x4)^2))+x5/(1+x6*exp(x7*7))+x8-4310 = 0,
	x1*exp(0-exp(x2-x3*(48-x4)^2))+x5/(1+x6*exp(x7*48))+x8-36955 = 0},
	{x1=10*(2*Generate(float)-1), x2=10*(2*Generate(float)-1), x3=10*(2*Generate(float)-1),
	x4=10*(2*Generate(float)-1), x5=10*(2*Generate(float)-1), x6=10*(2*Generate(float)-1),
	x7=10*(2*Generate(float)-1), x8=10*(2*Generate(float)-1)})

案例-12 结果：

| 求解器	参数值	方程误差值
1stOpt	第一组： x1: -1143.06680965009 x2: 1.74650544185396 x3: 0.211000039051278 x4: 2.21121626657505 x5: 37981.2000329656 x6: 5189.00679471742 x7: -0.961716225777624 x8: 116.86677668453 第二组： x1　876.60094572218 x2　-3.39977707570615 x3　-0.311164794133835 x4　1.89055245389685 x5　37700.2364638882 x6　3865.6159828945 x7　-0.913515606842387 x8　-745.23646388822	第一组： -2.98427949019242E-12 -1.90425453183707E-12 1.30739863379858E-12 2.8421709430404E-12 2.04636307898909E-12 -4.32009983342141E-12 1.81898940354586E-12 4.36557456851006E-11 第二组： 2.27373675443232E-13 -1.70530256582424E-12 -3.41060513164848E-13 -1.13686837721616E-12 9.09494701772928E-13 -9.09494701772928E-13 -8.18545231595635E-12 -3.63797880709171E-11
Matlab	无法给出正解	
Mathematica	无法给出正解	
Maple	无法给出正解	

　　本题难度较大，1stOpt 能以 80%的概率得到两组解，"3M"软件无一例外均以失败告终，即使给出很接近最优解的初值如：x=[9000, -2.5, 0.5, 6, 37000, 3800, -1, -9000]或 x=[-1200, 1.5, 0.3, 3.0, 35000, 6000, -0.5, 90]，仍然无法获得正确结果。

　　十二道方程组测试题计算对比结果汇总如下表及图。

测试题	正解成功率（%）			
	1stOpt	Matlab	Mathematica	Maple
1	90	0	0	0
2	50	0	0	0
3	100	0	0	100
4	90	0	0	0
5	90	0	0	0
6	100	0	0	0
7	100	0	0	0
8	100	0	0	100
9	100	10	0	0
10	100	0	0	50

11	90	0	0	0
12	80	0	0	0
平均（%）	90.83	0.83	0	20.83

成功率（%）
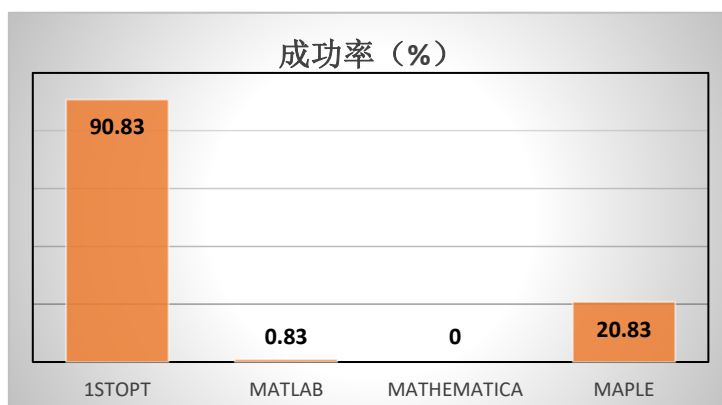
39.3 小结

基于前述计算对比结果，对这十二道测试题，整体而言，1stOpt 远远强于"3M"三剑客，而三剑客中，Maple 强于 Matlab，Matlab 又略强于 Mathematica 的零成功率。

或许会有疑问，这样比较是否合理？因为"3M"软件求解非线性方程组的算法基本都是类牛顿梯度算法的局部最优算法，这些算法就决定了计算求解是否收敛严重依赖于初始值的猜测的好坏，但合理初始值的猜测不要说对一般用户是难于上青天，即使顶尖专家或高手也经常无能为力，是极为棘手的事。在 1stOpt 出现之前，一般认为用户自己不断调整猜初始值是理所当然不得不做的"份内事"，因为算法理论上不可能保证任意初值的收敛，即使采用全局最优算法也无法做到。但 1stOpt 的出现及实际表现却表明"猜初值"的时代可以告一段落了，尤其对普通用户而言，只希望在应用软件中输入描述自己问题的代码，然后软件就能给出想要的正确结果，就这么简单质朴，至于软件本身在后台采用什么算法或方法，那应该是数学家和计算机科学家们的任务，从这一点而言 1stOpt 无疑是做的最好，使用简单，仅需关注问题本身而无需关注初始值、算法等。虽然 1stOpt 也无法保证百分之百从随机初始值求得正解，但相对于"3M"三剑客数学软件而言，已有了质的改变和进步。

Matlab、Mathematica 和 Maple 毫无疑问是当今市面上最伟大的三款数学科学计算软件，具有丰富众多的功能，总体综合能力"遥遥领先"，在此仅仅在非线性方程组求解方面进行了对比测试，"3M"软件中，采用不同的选项及代码构造方式会产生不同的结果，因此本文结论仅供参考。